2017 ACTA UNIVERSITATIS CAROLINAE PAG. 27–32
PHILOSOPHICA ET HISTORICA 2 / MISCELLANEA LOGICA

PRESERVING MEASURABILITY WITH COHEN ITERATIONS

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ABSTRACT
We describe a weak version of Laver indestructibility for a μ-tall cardinal κ, $\mu > \kappa^+$, where "weaker" means that the indestructibility refers only to the Cohen forcing at κ of a certain length. A special case of this construction is: if μ is equal to κ^{+n} for some $1 < n < \omega$, then one can get a model V^* where κ is measurable, and its measurability is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 \leq \alpha \leq \kappa^{+n}$ (Theorem 3.3).
Keywords: Cohen forcing, measurability
AMS subject code classification: 03E35, 03E55

1. Introduction

Assume κ is supercompact. In [7], Laver defined an iteration P of length κ such that in $V[P]$,[1] κ is still supercompact and every further κ-directed closed forcing preserves the supercompactness of κ (P is often called the *Laver preparation*). We also say that κ is Laver-indestructible in $V[P]$. The proof of this indestructibility result is essentially based on two useful properties of a supercompact cardinal κ in V: (i) for every $\mu \geq \kappa$, one can choose an elementary embedding $j : V \to M$ with critical point κ such that M is closed under μ-sequences existing in V; this closure is then used to find a *master condition* in M and proceed with a lifting argument which ensures that supercompactness is preserved,[2] (ii) there is a single function $f : \kappa \to V_\kappa$ such that for every $x \in V$, one can choose an embedding j in (i) so that $j(f)(\kappa) = x$ (this f is often called the *Laver function*).

A typical example of a κ-directed closed forcing is the Cohen forcing at κ, which we will denote by $\mathrm{Add}(\kappa, \alpha)$,[3] where α is any ordinal larger than 0. The fact that over $V[P]$, $\mathrm{Add}(\kappa, \alpha)$ preserves the measurability of κ is very useful when one wishes to use some

[1] $V[P]$ indicates a P-generic extension of V whenever it is not important to distinguish specific P-generic filters. For instance the statement "φ holds in $V[P]$" means that φ holds in $V[G]$ for every P-generic filter G.

[2] Assume $j : V \to M$ is an elementary embedding, P is a forcing notion, G is P-generic over V, and H is $j(P)$-generic over M. Then a sufficient condition for j to *lift*, i.e. a sufficient condition for the existence of $j^+ : V[G] \to M[H]$ with $j^+ \restriction V = j$, is that we have $j''G \subseteq H$. With supercompactness, we can often argue that $j''G$ is a condition in M (a master condition), and H can then be built below this master condition. For more details, see [3].

[3] Formally speaking, conditions in $\mathrm{Add}(\kappa, \alpha)$ are partial functions of size $< \kappa$ from $\kappa \times \alpha$ to 2. The ordering is by reverse inclusion.

https://doi.org/10.14712/24647055.2017.13

large cardinal properties of κ in $V[P][\mathrm{Add}(\kappa,\alpha)]$ (see for instance [4] where a model with the tree property at κ^{++}, κ strong limit singular with cofinality ω, is constructed starting with a supercompact κ).

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. As it turns out, it is the property (i) above which is more important: it is known that for instance a strong cardinal[4] κ has the analogue of the Laver function, but it is not known whether it can be made indestructible under κ-directed closed forcings.[5]

In this short paper we use the idea of Woodin (as described in [2]) to argue that it is possible to have a limited indestructibility of a μ-tall cardinal[6] κ, $\kappa^+ < \mu$ regular, in the sense that we can successively extend $V \subseteq V^1 \subseteq V^*$ so that forcing with $\mathrm{Add}(\kappa,\mu)$ over V^* yields the measurability of κ. See Section 2.

If $\mu = \kappa^{+n}$, $1 < n < \omega$, we can say more. If κ is $H(\kappa^{+n})$-hypermeasurable[7], V^* has the property that forcing with $\mathrm{Add}(\kappa,\alpha)$ over V^* for $0 < \alpha \le \kappa^{+n}$ yields the measurability, in fact hypermeasurability, of κ (Theorem 3.1 and Theorem 3.3). Note that in V^*, κ may actually stop being measurable[8] depending on the iteration P_κ which gives $V^* = V^1[P_\kappa]$; compare the constructions in Theorem 3.1 and 3.3.

Remark 1.1. We assume that the reader is familiar with the lifting arguments. The general reference is [3]; the more specific constructions used in the present paper are also given in [2].

2. Tall cardinals

In this section, we assume GCH. Let κ be μ-tall cardinal for some regular $\kappa^+ < \mu$.

Let $j : V \to M$ be a μ-tall embedding with the extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \;\&\; \alpha < \mu\}.$$

In particular, M is closed under κ-sequences in V and $\mu < j(\kappa) < \mu^+$. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have support κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has support which we denote ν, where $(\kappa^{++})^N < \nu < i(\kappa)$, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain ν.[9]

Let P denote the forcing $\mathrm{Add}(\kappa,\mu)$ in V, $Q = i(P)$, and let g be a Q-generic filter over V. Then the following hold:

4 A regular cardinal κ is *strong* if for every $\mu \ge \kappa$ there is $j : V \to M$ with critical point κ and $H(\mu) \subseteq M$.

5 A non-supercompact strong cardinal κ can be indestructible under κ-directed closed forcings by a method of [1], but κ needs to be supercompact in the ground model.

6 There is $j : V \to M$ with critical point κ such that M is closed under κ-sequences and $j(\kappa) > \mu$.

7 κ is *$H(\mu)$-hypermeasurable* (also *$H(\mu)$-strong*) if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$, $H(\mu) \subseteq M$, and M is closed under κ-sequences in V.

8 If in V^*, κ is not measurable, and it is measurable again in $V^*[\mathrm{Add}(\kappa,\alpha)]$ (for a specific α), it is more appropriate to call this step a "resurrection" of the measurability of κ.

9 ν needs to have the property that $k(\nu) \ge \mu$; some such ν always exists.

Theorem 2.1. *GCH. Forcing with Q preserves cofinalities and the following hold in $V[g]$:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.

(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.

(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.

(iv) g is Q-generic over $N[i^1(g)]$.

Proof. We show that Q is κ^+-closed and κ^{++}-cc in V. Closure is obvious by the fact that N is closed under κ-sequences in V. Regarding the chain condition, notice that every element of Q can be identified with the equivalence class of some function $f : \kappa \to \mathrm{Add}(\kappa, \mu)$. For $f, g : \kappa \to \mathrm{Add}(\kappa, \mu)$, set $f \leq g$ if for all $i < \kappa$, $f(i) \leq g(i)$; it suffices to check that the ordering $\leq$ on these f's is κ^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size κ^+ and is closed under κ-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most κ^+.

(i) and (ii). These follow by κ^+-distributivity of Q in V and the fact that j, i have support κ: the pointwise image of g generates a generic for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $\nu < i(\kappa^+)$, we use the distributivity of $i(Q)$ and the fact that k has support ν to argue that the pointwise image of $i^1(g)$ generates a generic filter which is equal to $j^1(g)$ by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. There are therefore mutually generic over N by Easton's lemma. □

Remark 2.2. It would be tempting to expect that j^1 is still $H(\mu)$-hypermeasurable if the original j was: however g is not included in $M[j^1(g)]$ and j^1 is therefore just μ-tall. There are some delicate issues involved if one wishes to preserve the $H(\mu)$-hypermeasurability of κ in Theorem 2.1. A natural strategy is to prepare below κ by a reverse Easton iteration. This approach is taken in [2] where it is also shown that if $\mu = \kappa^{++}$, then Q is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})$ and thus the preparation can be implemented by iterating $\mathrm{Add}(\alpha^+, \alpha^{++})$ at all inaccessible $\alpha \leq \kappa$. In [5], this representation is shown for $\mu = \kappa^{+n}$ for $2 \leq n < \omega$, i.e. $i(\mathrm{Add}(\kappa, \kappa^{+n}))$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{+n})$. It seems it is possible to continue up to the first cardinal above κ with cofinality κ, but it is unclear whether it can be extended further.

Remark 2.3. The loss of the $H(\mu)$-hypermeasurability of j^1 may prevent the use of this method in more complicated situations (such as a subsequent definition of Radin forcing to achieve results of a more global character).

Let us work in the model $V[g] = V^1$ and let us use the notation $j^1, i^1, k^1, V^1, M^1, N^1$ to denote the resulting models and embeddings in Theorem 2.1. Using a fast-function forcing of Woodin, we can assume that there is $f : \kappa \to \kappa$ in V such that $j(f)(\kappa) = \mu$. Let us denote $f(\alpha)$ by μ_α; let $C(f)$ denote the closed unbounded set of the closure points of f: if $\alpha \in C(f)$, then for all $\beta < \alpha$, $f(\beta) < \alpha$.

Theorem 2.4. *There is a forcing iteration R_κ defined in V^1 such that*

$$V^1[R_\kappa][\mathrm{Add}(\kappa, \mu)] \vDash \kappa \text{ is } \mu\text{-tall},$$

where $\mathrm{Add}(\kappa, \mu)$ *is defined in* $V[R_\kappa]$.

Proof. Define R_κ to be the following Easton-supported iteration:

$$R_\kappa = \langle (R_\alpha, \dot{Q}_\alpha) \mid \alpha \in C(f), \alpha \text{ inaccessible} \rangle, \tag{2.1}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha, \mu_\alpha)$.

The proof uses the usual surgery argument (see [3]) with Fact 2.5 which allows us to use the generic filter g added in V^1 (for the i^1-image of $\mathrm{Add}(\kappa, \mu)^{V^1}$) in the model $V^1[R_\kappa]$ (for the proof, see Fact 2 in [2]).[10]

Fact 2.5. *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa, \mu)^{V[S]}/S$ is isomorphic to* $\mathrm{Add}(\kappa, \mu)$.

Now we proceed with the proof of Theorem 2.4. Let $G_\kappa * H$ be $R_\kappa * \mathrm{Add}(\kappa, \mu)^{V^1[R_\kappa]}$-generic over V^1. Using the standard methods, lift[11] in $V^1[G_\kappa * H]$ the embeddings j^1, i^1, k^1 to R_κ, obtaining commutative triangle $j^1 : V^1[G_\kappa] \to M^1[j^1(G_\kappa)]$, $i^1 : V^1[G_\kappa] \to N^1[i^1(G_\kappa)]$, and $k^1 : N^1[i^1(G_\kappa)] \to M^1[j^1(G_\kappa)]$.

Using the elementarity of i^1, Fact 2.5 applied with $S = i^1(R_\kappa)$ and $i^1(\mathrm{Add}(\kappa, \mu))$ shows that g – which is present in V^1 – yields a generic filter g' for the forcing $i^1(\mathrm{Add}(\kappa, \mu))$ of $N^1[i^1(G_\kappa)]$. The pointwise image of g' via k^1 generates a $j^1(\mathrm{Add}(\kappa, \mu))$-generic filter over $M^1[j^1(G_\kappa)]$, which is then modified by the standard surgery argument to allow for lifting j^1 to $V^1[G_\kappa * H]$ (for details see [2]); i.e. if we denote the lifting of j^1 by j^2, then

$$j^2 : V^1[G_\kappa][H] \to M^1[j^1(G_\kappa * H)]$$

witnesses the measurability, and in fact μ-tallness, of κ. □

3. Hypermeasurable cardinals

It seems natural to extend Theorem 2.4 and have that the measurability of κ ensured by $\mathrm{Add}(\kappa, \alpha)$ for any ordinal α, $0 < \alpha \leq \mu$. We will show that this can be achieved with some additional assumptions on μ. For concreteness, we will focus on the example where $\mu = \kappa^{+n}$ for some $1 < n < \omega$.

First, in Theorem 3.1, we provide a standard construction which actually forces κ to stop being measurable in V^*; the measurability of κ is then resurrected by $\mathrm{Add}(\kappa, \alpha)$ for any $\kappa^+ \leq \alpha \leq \kappa^{+n}$.

Theorem 3.1. *(GCH) Let $1 < n < \omega$ be fixed and assume κ is $H(\kappa^{+n})$-hypermeasurable. Then there is an iteration P^1 such that in $V[P^1] = V^1$, κ is still κ^{+n}-hypermeasurable, and for some reverse Easton iteration P_κ defined in V^1, κ stops being measurable in $V^* = V^1[P_\kappa]$. In V^*, the measurability – in fact the hypermeasurability – of κ is resurrected by Cohen forcing* $\mathrm{Add}(\kappa, \alpha)$ *for any $\kappa^+ \leq \alpha \leq \kappa^{+n}$.*

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[10] Recall that Q_μ – mentioned in Fact 2.5 – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa, \mu)$ of $V[S]$. The ordering is $\tau \leq \sigma \leftrightarrow 1_S \Vdash \tau \leq \sigma$.

[11] For simplicity, we use the notation j^1, i^1, k^1 to denote the partial liftings of the embeddings j^1, i^1, k^1.

Proof. Let j be an extender embedding witnessing the $H(\kappa^{+n})$-hyper-measurability of κ, and let i be a normal embedding generated by the normal measure U derived from j. Recall Lemma 3.2 from [5] which implies that if $i : V \to N$ is an embedding generated by a normal measure on κ, then

$$\text{Add}(i(\kappa), i(\kappa)^{+n})^N \cong \text{Add}(\kappa^+, \kappa^{+n}). \tag{3.2}$$

Define P^1 is an Easton-supported iteration

$$\langle (P^1_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa, \alpha \text{ is inaccessible}\rangle * \dot{Q}_\kappa,$$

where for an inaccessible $\beta \leq \kappa$, $\dot{Q}_\beta$ is $\text{Add}(\beta^+, \beta^{+n})$ of $V[P^1_\beta]$.

Let $G_\kappa * g$ be $P^1_\kappa * \dot{Q}_\kappa$-generic over V, and denote $V[G_\kappa * g]$ by V^1. Let j^1 and i^1 be the liftings of j and i.

In V^1 define P_κ as an Easton supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \text{ is inaccessible}\rangle, \tag{3.3}$$

where $\dot{Q}_\alpha$ denotes the forcing $\text{Add}(\alpha, \alpha^{+n})$ of $V^1[P_\alpha]$.

First note that κ stops being measurable in $V^* = V^1[P_\kappa]$ by the application of the gap-forcing theorem of [6]: a hypothetical embedding k with critical point κ found in V^* could be written as an embedding from $V^1[P_\kappa]$ to some $N[j(P_\kappa)]$, with $N \subseteq V^1$; in particular a generic filter for $j(P_\kappa)$ would need to add a non-trivial generic filter at stage κ which cannot be found in $V^1[P_\kappa]$.

The rest of the Theorem follows from the following Claim:

Claim 3.2. *Let α be an ordinal, $\kappa^+ \leq \alpha \leq \kappa^{+n}$. Then κ is still measurable in $V^1[P_\kappa]$ $[\text{Add}(\kappa, \alpha)]$, where $\text{Add}(\kappa, \alpha)$ is defined in $V^1[P_\kappa]$.*

Proof. It suffices to show the Claim for α's which are cardinals. So assume $\kappa^{+m} = |\alpha|$ for some $1 \leq m \leq n$. Choose in V^1 an embedding $j_m : V^1 \to M_m$ which witnesses the $H(\kappa^{+m})$-hypermeasurability of κ with $\kappa^{+m} < j_m(\kappa) < \kappa^{+m+1}$ (this is possible since $2^\kappa = \kappa^+$ in V^1). By the definition of P_κ, $j_m(P_\kappa)(\kappa)$ is equal to $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. Since $(\kappa^{+n})^{M_m}$ has size κ^{+m} in V^1, $\text{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ is equivalent to $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$, and therefore the generic for $\text{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ provides a generic for $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. The argument is then finished as in Theorem 2.4, using the fact that the generic g for $i^1(\text{Add}(\kappa, \kappa^{+n}))$ is also generic for $i^1(\text{Add}(\kappa, \kappa^{+m}))$. □

This concludes the proof of Theorem 3.1. □

Note that the method in the proof of Theorem 3.1 does not work for the case of α smaller than κ^+: every elementary embedding $k : V^1 \to M$ with critical point κ sends κ above κ^+ and therefore $\kappa^+ \leq |\kappa^{+n}|$ in V^1; thus $k(P_\kappa)(\kappa)$, which is $\text{Add}(\kappa, \kappa^{+n})^{M[P_\kappa]}$, is in $V^1[P_\kappa]$ equivalent to the Cohen forcing at κ of length at least κ^+. It follows that to lift the embedding, we need to force over $V^1[P_\kappa]$ with a Cohen forcing at κ of length at least κ^+. If $\alpha < \kappa^+$, this condition is not satisfied. We remedy this by a more complicated construction in Theorem 3.3.

Theorem 3.3. *With the assumptions and the notation as in Theorem 3.1, one can define P_κ so that κ is measurable in V^*, and its measurability – in fact hypermeasurability – is indestructible by $\text{Add}(\kappa, \alpha)$ for any $0 < \alpha \leq \kappa^{+n}$.*

Proof. Modify the definition of P_κ in (3.3) so that at an inaccessible $\alpha < \kappa$, $\dot{Q}_\alpha$ is chosen generically[12] amongst the following forcings: $\{1\}$ (the trivial forcing), and $\mathrm{Add}(\alpha, \alpha^{+k})$, for $0 \le k \le m$.

Then one can argue that κ is still measurable in V^*: while lifting the embedding j^1, it suffices to work below a condition in $j^1(P_\kappa)$ which chooses the trivial forcing $\{1\}$ at stage κ.

To argue that for any $0 < \alpha \le \kappa^{+n}$, κ is still measurable in $V^*[\mathrm{Add}(\kappa, \alpha)]$, work below a condition in $j^1(P_\kappa)$ which chooses the right forcing at stage κ. □

4. Open questions

Q1. Is it possible to generalise Theorem 2.4 so that μ is still $H(\mu)$-hypermeasurable if the original embedding j was $H(\mu)$-hypermeasurable? This would require some sort of preparation below κ in the model V^1 (analogously to the methods in Theorem 3.1).

A related question is this:

Q2. Is it possible to characterise the forcings $i(\mathrm{Add}(\kappa, \mu))$, where $i : V \to N$ is a normal measure ultrapower as in Theorem 2.1? We know that this forcing does not collapse (it is κ^+-closed and κ^{++}-cc in V), but does it have a uniform representation? In particular, is it isomorphic to $\mathrm{Add}(\kappa^+, \mu)$ of V?

Acknowledgments

The work was mainly supported by FWF/GAČR grant I 1921-N25.

A support of travel grant Mobility 7AMB15AT035 is also acknowledged.

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12 The "lottery preparation" in the terminology of Hamkins.